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# CONSTRUCTION OF CONTROL WITH CONSTRAINTS FOR NONLINEAR SYSTEMS WITH COEFFICIENTS DEPENDING ON THE CONTROL OBJECT STATE

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*We consider an optimal control problem on a finite time-interval for a three-sector economic control object. We reduce the economic system to an optimal control problem for a nonlinear system with coefficients independent of the control object state and find a nonlinear synthesizing control based on the feedback principle and certain constraints on control. The results obtained for the nonlinear system are used to construct the control parameters in the mathematical model of a three-sector economic control object. We find an optimal distribution between the labor and investment resources satisfying the balance relations. Bibliography: 3 titles. Illustrations: 2 figures.*

In [1, 2], Lagrange multipliers are used to study optimal control problems for technical systems and the linearized system of an economic cluster. In this paper, we consider an economic system and transform it to an optimal control problem for a class of nonlinear systems with coefficients depending on the control object state.

We transform the nonlinear differential equation describing the original control system to a system of linear structure, but with parameters depending on the state. Synthesizing the control, we use the nonlinear quadratic functional which allows us to construct the Riccati matrix equation with parameters independent of the control object state. This approach is a basis of the synthesis of nonlinear optimal control systems. We propose a combined method based on constructing a nonlinear feedback, which allows us to represent the sought control as a synthesizing control depending on the state of the nonlinear system and current time. Moreover, owing to this method, it is possible to take into account constraints on the control values.

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The results obtained for nonlinear systems are used to construct the control parameters of a three-sector economic control object on a finite time interval. We note that the share of labor and investment resources are variable in all the three economy sectors.

# 1 The Three-Sector Economic Control Object Model

We consider the optimal control problem for a three-sector economic control object model consisting of the following sectors: $i = 0$ (material), $i = 1$ (fund-creator), $i = 2$ (consumer). The mathematical model consists of the following three components [3]:

(a) three differential equations governing the dynamics of the capital-labor ratios

$$\dot{k}_i = -\lambda_i k_i + (s_i/\theta_i)x_1, \quad k_i(0) = k_i^0, \quad \lambda_i > 0, \quad i = 0, 1, 2, \tag{1.1}$$

(b) three functions of Cobb–Douglas type specific output

$$x_i = \theta_i A_i k_i^{\alpha_i}, \quad A_i > 0, \quad 0 < \alpha_i < 1, \quad i = 0, 1, 2, \tag{1.2}$$

(c) three balance relations

$$s_0 + s_1 + s_2 = 1, \quad s_0 \geqslant 0, \quad s_1 \geqslant 0, \quad s_2 \geqslant 0, \tag{1.3}$$

$$\theta_0 + \theta_1 + \theta_2 = 1, \quad \theta_0 \geqslant 0, \quad \theta_1 \geqslant 0, \quad \theta_2 \geqslant 0, \tag{1.4}$$

$$(1 - \beta_0)x_0 = \beta_1 x_1 + \beta_2 x_2, \quad \beta_0 \geqslant 0, \quad \beta_1 \geqslant 0, \quad \beta_2 \geqslant 0. \tag{1.5}$$

Here, the state of the economic system (the capital-labor ratio) is described by the vector $(k_0, k_1, k_2)$, $(s_0, s_1, s_2, \theta_0, \theta_1, \theta_2)$ is the control vector, $(s_0, s_1, s_2)$ indicates the share of sectors in the distribution of investment resources, $(\theta_0, \theta_1, \theta_2)$ indicates the share of sectors in the distribution of labor resources, $x_i$ is the specific output (the number of products produced in the $i$th sector per worker), $\beta_i$ is the direct material cost for production in the $i$th sector, $i = 0, 1, 2$. The initial state of the system is characterized by $k_0^0$, $k_1^0$, $k_2^0$, where $k_i^0 = k_i(0)$ is the capital-labor ratio for the $i$th sector, $i = 0, 1, 2$, at $t = 0$. Our goal is to transfer the nonlinear system from its initial state to a desired state in a time interval $[0, T]$. For the desired final state $(k_0^s, k_1^s, k_2^s)$ we take the equilibrium state of the system defined in [3] as follows:

$$k_1^s = \left(\frac{s_1 A_1}{\lambda_1}\right)^{\frac{1}{1-\alpha_1}}, \quad k_0^s = \frac{s_0\theta_1 A_1 (k_1^s)^{\alpha_1}}{\lambda_0\theta_0}, \quad k_2^s = \frac{s_2\theta_1 A_1 (k_1^s)^{\alpha_1}}{\lambda_2\theta_2}, \tag{1.6}$$

where $k_i^s$, $i = 0, 1, 2$, depend on the control vector $(s_0, s_1, s_2, \theta_0, \theta_1, \theta_2)$ whose stationary values $(s_0^s, s_1^s, s_2^s, \theta_0^s, \theta_1^s, \theta_2^s)$ were obtained in [3].

# 2 Statement of the Problem with Constraints on Control

We write the mathematical model (1.1) in the form

$$\dot{y}(t) = Ay(t) + BD(y)u(t) + B(D(y) - D(k^s))v^s, \quad y(t_0) = y_0, \quad t \in [t_0, T], \tag{2.1}$$

where

$$y_1 = k_1 - k_1^s, \quad y_2 = k_2 - k_2^s, \quad y_3 = k_0 - k_0^s,$$

$$u_1 = s_1 - v_1^s, \quad u_2 = \frac{s_2\theta_1}{\theta_2} - v_2^s, \quad u_3 = \frac{s_0\theta_1}{\theta_0} - v_3^s, \quad v_1^s = s_1^s, \quad \frac{s_2^s\theta_1^s}{\theta_2^s} = v_2^s, \quad \frac{s_0^s\theta_1^s}{\theta_0^s} = v_3^s,$$

$$f_1(y_1) = (y_1 + k_1^s)^{\alpha_1}, \quad f_2(y_2) = (y_2 + k_2^s)^{\alpha_2}, \quad f_3(y_3) = (y_3 + k_0^s)^{\alpha_0},$$

$$A = \begin{pmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_2 & 0 \\ 0 & 0 & -\lambda_0 \end{pmatrix}, \quad B = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_1 \end{pmatrix},$$

$$D(y) = \begin{pmatrix} (y_1 + k_1^s)^{\alpha_1} & 0 & 0 \\ 0 & (y_1 + k_1^s)^{\alpha_1} & 0 \\ 0 & 0 & (y_1 + k_1^s)^{\alpha_1} \end{pmatrix}, \quad D(k^s) = \begin{pmatrix} (k_1^s)^{\alpha_1} & 0 & 0 \\ 0 & (k_1^s)^{\alpha_1} & 0 \\ 0 & 0 & (k_1^s)^{\alpha_1} \end{pmatrix},$$

The constants $k^s$ and $v^s$ are found from (1.6) in the stationary case and satisfy the algebraic equation

$$Ak^s + BD(k^s)v^s = 0, \tag{2.2}$$

where $y = (y_1, y_2, y_3)^*$ is the object state vector and $u = (u_1, u_2, u_3)^*$ is the control vector.

Using (2.1) and (1.3)–(1.5), we write the control object in the form

$$\dot{y}(t) = Ay(t) + BD(y)v(t), \quad y(t_0) = y_0, \quad t \in [t_0, T], \tag{2.3}$$

where

$$v(t) \in V(t) = \{v|\gamma_1(t) \leqslant v(t) - (E - D^{-1}(y)D(k^s))v^s \leqslant \gamma_2(t), \quad t \in [t_0, T], \ \gamma_1, \gamma_2 \in C[t_0, T]\},$$

$$u(t) = v(t) - (E - D^{-1}(y)D(k^s))v^s, \quad g(u, y, s, \theta) = 0.$$

Let the system (2.3) be controllable. The matrices $A$ and $B$ satisfy the controllability condition, i.e., Rank $[B, AB, \ldots, A^{n-1}B] = n$.

We consider the following functional depending on the control and state of the object:

$$J(u) = \frac{1}{2}\int_{t_0}^{T} [y^*(t)Q(y)y(t) + v^*(t)Rv(t)]dt + \frac{1}{2}y^*(T)Fy(T), \tag{2.4}$$

where $Q(y) = KBD(y)R^{-1}D^*(y)B^*K - KBD(k^s)R^{-1}D^*(k^s)B^*K + Q_1$ is a positive semidefinite matrix and $R$, $D(y)$, $F$ are positive definite matrices.

We study the problem of finding a synthesizing control $v(y, t)$ moving the system (2.1) from the initial state $y(t_0) = y_0$ to the desired equilibrium state $y(T) = 0$ in the time interval $[t_0, T]$ and minimizing the functional (2.4).

## 3 Solution of the Problem with Constraints on Control

To the expression for the functional (2.4) we add the system of differential equations (2.3) with factor $\lambda = Ky + q(t)$ and the expression

$$\lambda_1^*(t)[\gamma_1 - u(t)] + \lambda_2^*(t)[u(t) - \gamma_2] + \lambda_3^*(t)[y(t) - W(t, T)q(t)], \tag{3.1}$$

where $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 0$. As a result, we obtain the functional

$$L(y,v) = \int_{t_0}^{T} \Big\{ \frac{1}{2} y^*(t) Q(y) y(t) + \frac{1}{2} v^*(t) R v(t) + (Ky + q(t))^* (Ay + BD(y)v(t) - \dot{y})$$
$$+ \lambda_1^*(t)[\gamma_1 - u(t)] + \lambda_2^*(t)[u(t) - \gamma_2] + \lambda_3^*(t)[y(t) - W(t,T)q(t)] \Big\} dt$$
$$+ \frac{1}{2} y^*(T) F y(T), \tag{3.2}$$

where $q(t)$ is a vector and $K$ is a constant symmetric positive definite $(n \times n)$-matrix.

For the problem under consideration the release principle is to reduce the original optimal control problem with constraints to an optimal control problem without constraints, but possessing a solution coinciding with the solution to the original problem [1, 2]. We set

$$V(y,t) = \frac{1}{2} y^* K y + y^* q(t), \tag{3.3}$$

$$M(y,v,t) = \frac{1}{2} y^* Q(y) y + \frac{1}{2} v^* R v + (Ky + q(t))^* (Ay(t) + BD(y)v(t)) + y^* \dot{q}(t)$$
$$+ \lambda_1^*(t)[\gamma_1 - u(t)] + \lambda_2^*(t)[u(t) - \gamma_2] + \lambda_3^*(t)[y(t) - W(t,T)q(t)]. \tag{3.4}$$

Then the functional (3.2) can be written as

$$L(y,u) = V(y_0,t_0) + \int_{t_0}^{T} M(y,u,t)dt - V(y(T),T) + \frac{1}{2} y^*(T) F y(T). \tag{3.5}$$

The sought control is found from the equality

$$R(u + (E - D^{-1}(y)D_s)v_s) = -D^* B^* (Ky + q(t)) - (\lambda_2 - \lambda_1), \tag{3.6}$$

where the matrices $K$, $W(t,T)$ and vector $q(t)$ satisfy the differential equations

$$KA + A^*K - KBD(k^s)R^{-1}D^*(k^s)B^*K + Q_1 = 0, \tag{3.7}$$

$$\dot{W} = WA_1^*(y,t) + A_1(y,t)W - B_1(y), \quad W(T,T) = (F-K)^{-1}, \quad F \gg K, \tag{3.8}$$

$$\dot{q} = -A_1^*(y,t)q + W^{-1}(t,T)BD\varphi(y,t), \quad q(T) = (F-K)y(T) \tag{3.9}$$

for $t \in [t_0, T]$, where

$$\begin{aligned} &A_1(y,t) = A - B_1(y)K(t), \quad B_1(y) = BD(y)R^{-1}D^*(y)B^*, \\ &\varphi(y,t) = R^{-1}[\lambda_1(y,t) - \lambda_2(y,t)], \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\lambda_1(y,t) = R\max\{0; \gamma_1 - \omega(y,t)\} \geqslant 0, \quad \lambda_2(y,t) = R\max\{0; \omega(y,t) - \gamma_2\} \geqslant 0, \\ &\omega(y,t) = -(E - D^{-1}(y)D_s)v_s - R^{-1}D^*(y)B^*(Ky + q(t)), \quad D_s = D(k^s). \end{aligned} \tag{3.11}$$

Assume that Equations (3.7), (3.8) are solvable. Then the differential equations defining the motion law for the system can be written as

$$\dot{y} = A_1(y,t)y(t) - BD(y)R^{-1}D^*(y)B^*q(t) + BD(y)\varphi(y,t), \quad y(t_0) = y_0. \tag{3.12}$$

We note that the initial condition for (3.9) is obtained from the equality

$$y(t) = W(t,T)q(t), \quad t \in [t_0, T]. \tag{3.13}$$

We formulate the results for the optimal control problem (2.1)–(2.4).

**Theorem 3.1.** *Assume that $Q(y)$ is a positive semidefinite matrix, $R$, $F$, $D(y)$ are positive definite matrices for $t_0 \leqslant t \leqslant T$, and $W_0 = W(t_0,T)$ is a positive definite matrix. We assume that the system (2.3) is completely controllable at $t_0$. Then for the optimality of $(y(t), u(t))$ in the problem (2.3), (2.4) the following conditions are sufficient*:

1) *$y(t)$ satisfies the differential equation*

$$\dot{y} = A_1(y,t)y(t) - B_1(y)q(t) + BD(y)\varphi(y,t), \quad y(t_0) = y_0, \tag{3.14}$$

2) *the control is defined by*

$$u(y,t) = -(E - D^{-1}(y)D_s)v_s) - R^{-1}D^*(y)B^*(Ky + q(t)) + \varphi(y,t). \tag{3.15}$$

The matrices $K$ and $W(t,T)$ are solutions to Equations (3.7) and (3.8), the function $q(t)$ satisfies the differential equation (3.9), and the vector-valued function $\varphi(y(t),t)$ is defined by (3.10).

# 4 Algorithm

We describe an algorithm for solving the optimal control problem (2.1)–(2.4). This algorithm is convenient for implementation on a computer.

1. Solve the system of algebraic and differential equations (3.7) and (3.8) to obtain the matrices $K$ and $W(t,T)$ on $[t_0,T]$.

2. Impose the condition $y(t_0) = y_0$ and compute $q(t_0) = W^{-1}(t_0,T)y(t_0)$.

3. Integrate the system (3.12), (3.9) on $[t_0,T]$ with the initial conditions $y(t_0) = y_0$, $q(t_0) = W^{-1}(t_0,T)y(t_0)$ and print the graphs of optimal trajectory $y(t)$ and optimal control $u(t)$ .

4. Under the assumption that the state $y(t)$ and optimal control $u(t)$ are already found, we have

$$\begin{aligned}
&u(t) = v(t) - (E - D^{-1}(y)D(k^s))v^s,\\
&f_1(y_1) = (y_1 + k_1^s)^{\alpha_1}, \quad f_2(y_2) = (y_2 + k_2^s)^{\alpha_2}, \quad f_3(y_3) = (y_3 + k_0^s)^{\alpha_0},\\
&\xi = \frac{\beta_1 A_1 f_1(y_1) + \beta_2 A_2 f_2(y_2)(1 - u_1 - v_1^s)/(u_2 + v_2^s)}{(1-\beta_0)A_0 f_3(y_3)(1 - u_1 - v_1^s)/(u_3 + v_3^s) + \beta_2 A_2 f_2(y_2)(1 - u_1 - v_1^s)/(u_2 + v_2^s)},
\end{aligned} \tag{4.1}$$

which implies (1.5),

$$s_1 = u_1 + v_1^s, \quad s_2 = (1-\xi)(1 - u_1 - v_1^s), \quad s_0 = \xi(1 - u_1 - v_1^s), \tag{4.2}$$

which implies (1.3),

$$\theta_1 = \frac{1}{1 + s_0/(u_3 + v_3^s) + s_2/(u_2 + v_2^s)}, \quad \theta_2 = \frac{(1-\xi)(1-s_1)\theta_1}{(u_2 + v_2^s)}, \quad \theta_0 = \frac{\xi(1-s_1)\theta_1}{(u_3 + v_3^s)}, \tag{4.3}$$

which implies (1.4).

# 5 Numerical Experiment of Finding Optimal Distribution of Labor and Investment Resources

The numerical experiment was performed for the following values of parameters:

| $i$ | $\alpha_i$ | $\beta_i$ | $\lambda_i$ | $A_i$ | $s_i^s$ | $\theta_i^s$ | $k_i^s$ |
|---|---|---|---|---|---|---|---|
| 0 | 0.46 | 0.39 | 0.05 | 6.19 | 0.2763 | 0.3944 | 966.4430 |
| 1 | 0.68 | 0.29 | 0.05 | 1.35 | 0.4476 | 0.2562 | 2410.1455 |
| 2 | 0.49 | 0.52 | 0.05 | 2.71 | 0.2761 | 0.3494 | 1090.1238 |

We solve the optimal control problem with the initial state

$$y(t_0) = (-700, \quad -300, \quad 300)^*, \tag{5.1}$$

and matrices

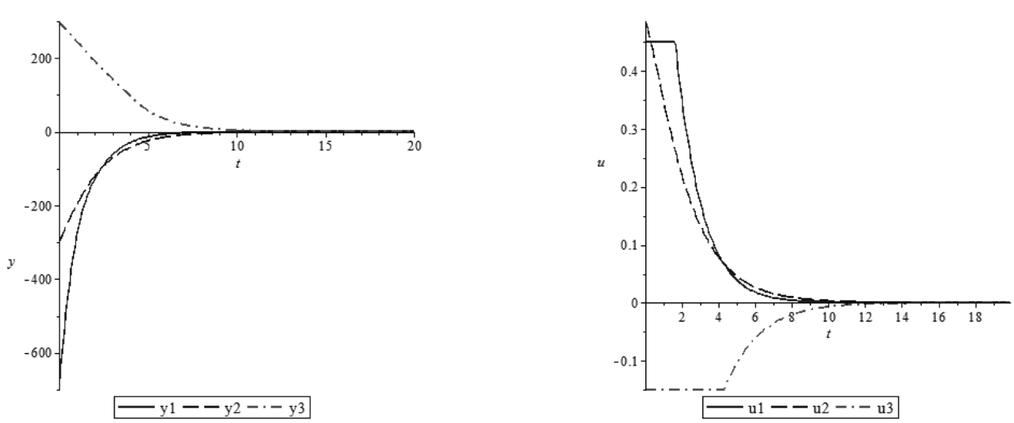
$$R = \begin{pmatrix} 200 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 200 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 16 \cdot 10^{-4} & 0 & 0 \\ 0 & 8 \cdot 10^{-4} & 0 \\ 0 & 0 & 8 \cdot 10^{-4} \end{pmatrix},$$

$$K = \begin{pmatrix} 0.1968 \cdot 10^{-2} & 0 & 0 \\ 0 & 0.1354 \cdot 10^{-2} & 0 \\ 0 & 0 & 0.1354 \cdot 10^{-2} \end{pmatrix}.$$

The results of calculations of the state of the system are presented in Figure 1 (a). Figure 1 (b) shows that the optimal controls do not go beyond the domain $V$ defined by the constraints. In the case under consideration, the constraints are given by

$$-0.45 \leqslant u_1 \leqslant 0.45, \quad -0.1 \leqslant u_2 \leqslant 0.8, \quad -0.15 \leqslant u_3 \leqslant 0.75. \tag{5.2}$$

Here, the control components $u_1(t)$ and $u_3(t)$ lie on the boundary of $V$ during the time intervals $[0, t_1]$ and $[0, t_2]$ respectively, but for $t \in [t_1, T]$ and $t \in [t_2, T]$, they lie inside the domain $V$. The controls $u_1(t)$ and $u_3(t)$ are switched at the times $t_1 = 1.553$ and $t_2 = 4.314$ respectively.




(a) (b)

Figure 1. The graphs of trajectories $y(t)$ (a) and the optimal control $u(t)$ (b).

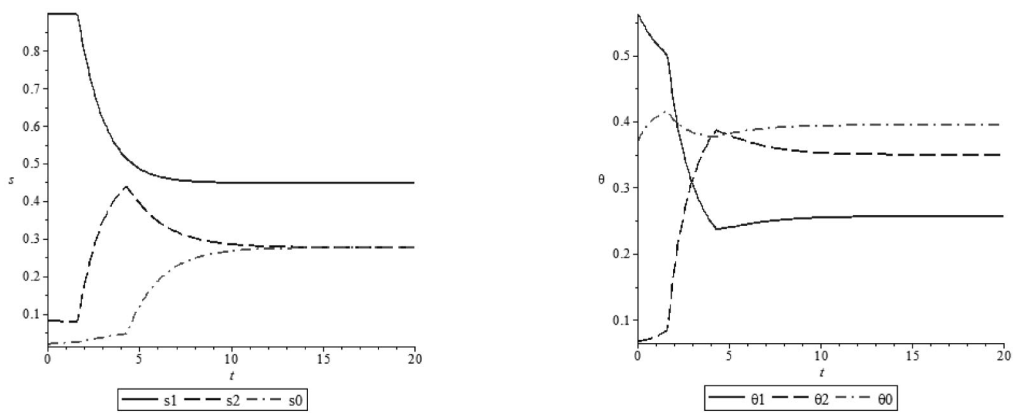
We write the optimal values of the state at a finite time for $T = 20$:

$$y_1(T) = -0.7292 \cdot 10^{-4}, \quad y_2(T) = -0.2731 \cdot 10^{-2}, \quad y_3(T) = 0.6313 \cdot 10^{-2}$$

and the optimal values of the control at a finite time for $T = 20$:

$$u_1(T) = 6.5097 \cdot 10^{-7}, \quad u_2(T) = 0.2432 \cdot 10^{-4}, \quad u_3(T) = -0.5620 \cdot 10^{-4}.$$

Using (4.1)–(4.3), we find the optimal distribution of the labor $(\theta_1(t),\ \theta_2(t),\ \theta_0(t))$ and investment $(s_1(t),\ s_2(t),\ s_0(t))$ resources. Figure 2 shows changes in the resources satisfying the balance relations (1.3)–(1.5).




(a) (b)

Figure 2. The graphs of the optimal distribution of investment (a) and labor (b) resources satisfying the balance relations (1.3)–(1.5).

The values of the investment $(s_1(t),\ s_2(t),\ s_0(t))$ and the labor resources $(\theta_1(t),\ \theta_2(t),\ \theta_0(t))$ at a finite time for $T = 20$ converge to the stationary state with the approximation estimates

$$|s_1(T) - s_1^s| = 0.6510 \cdot 10^{-6}, \quad |s_2(T) - s_2^s| = 0.1017 \cdot 10^{-3}, \quad |s_0(T) - s_0^s| = 0.1024 \cdot 10^{-3},$$

$$|\theta_1(T) - \theta_1^s| = 0.1430 \cdot 10^{-4}, \quad |\theta_2(T) - \theta_2^s| = 0.2386 \cdot 10^{-4}, \quad |\theta_0(T) - \theta_0^s| = 0.3816 \cdot 10^{-4}.$$

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